Closing Thu: TN 2, TN 3

TN 2 & 3: Higher order approx.

Recall*: 1st Taylor polynomial*

$$T_1(x) = f(b) + f'(b)(x-b)$$

Error Bound

On interval [a,b], if $|f''(x)| \leq M$,

then $|f(x) - T_1(x)| \leq \frac{M}{2}|x-b|^2$.

Entry Task: Let $f(x) = x^{1/3}$.

(a) Find the 1st Taylor Polynomial based at $b = 8$.

(b) Give a bound on the error over the interval $[7,9]$.

2nd Taylor Polynomial is given by

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2} f''(b)(x-b)^2$$

Quadratic error bound theorem

On interval [a,b], if $|f'''(x)| \leq M$,

then $|f(x) - T_2(x)| \leq \frac{M}{6}|x-b|^3$.

Example:

Find the 2nd Taylor polynomial for $f(x) = x^{1/3}$ based at b = 8 and find an error bound on the interval [7,9].

Taylor Approximation Idea:
If two functions have ***all*** the same derivative values, then they are the same function (up to a constant).
Let's compare derivatives of $f(x)$ and $T_2(x)$ at b.

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$$
$$T_2'(x) = 0 + f'(b) + f''(b)(x-b)$$
$$T_2''(x) = 0 + 0 + f''(b)$$
$$T_2'''(x) = 0$$

Now plug in x = b to each of these.

- What do you see?
- Why did we need a ½ ?
- What would $T_3(x)$ look like?
- What would $T_4(x)$ look like?
  ($T_5(x)$?, $T_6(x)$?...)

n^th Taylor polynomial

$$f(b)+f'(b)(x-b)+\frac{1}{2}f''(b)(x-b)^2+\frac{1}{3!}f'''(b)(x-b)^3+\cdots+\frac{1}{n!}f^{(n)}(b)(x-b)^n$$

In sigma notation:

$$T_n(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(b)(x-b)^k$$

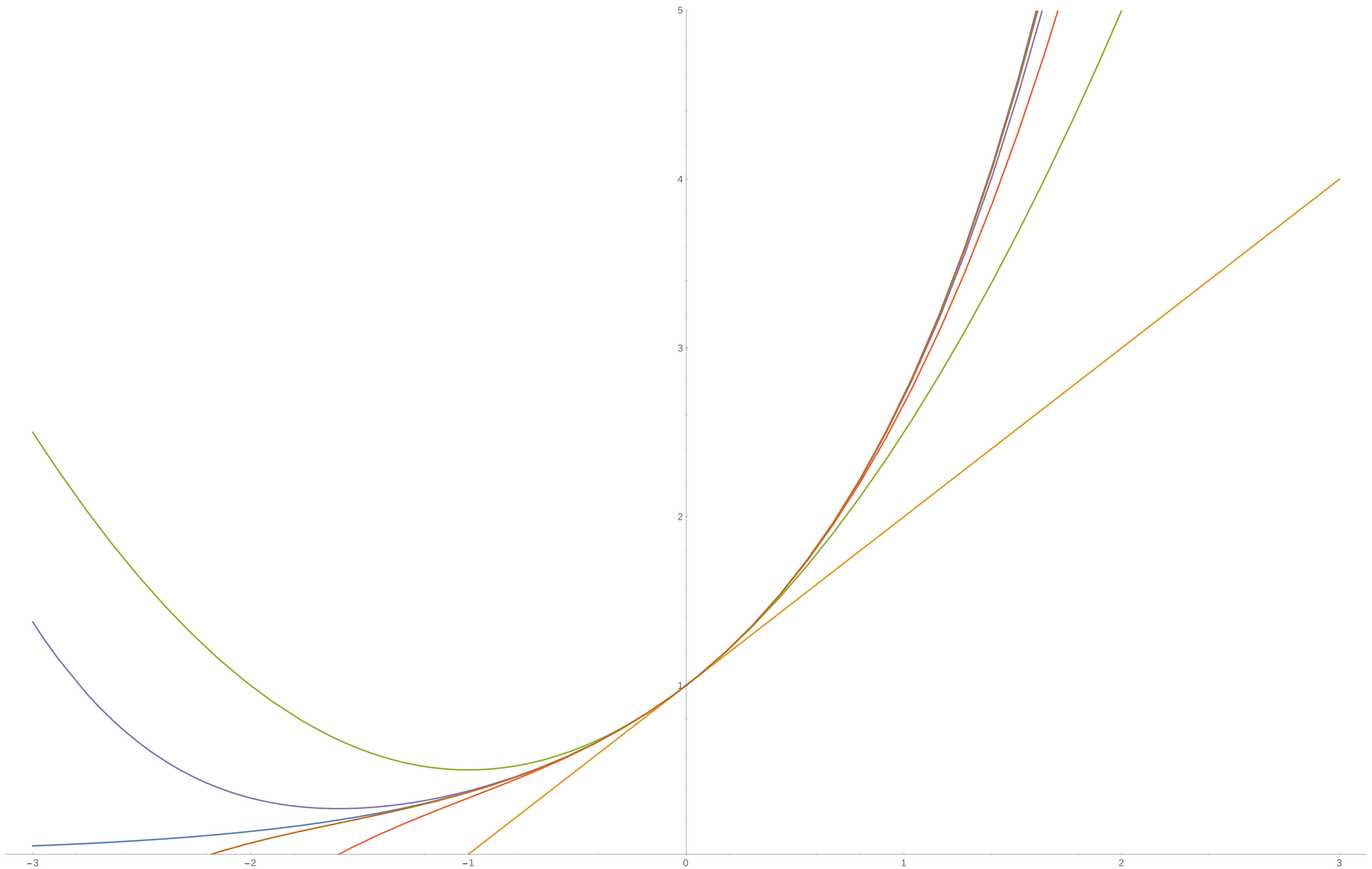
Example: Find the 9^th Taylor polynomial for $f(x) = e^x$ based at $b = 0$, and give an error bound on the interval $[-2,2]$.

Taylor's Inequality (error bound):
on a given interval [a,b],
if $|f^{(n+1)}(x)| \le M$, then

$$|f(x)-T_n(x)|\le\frac{M}{(n+1)!}|x-b|^{n+1}$$

$f(x) = e^x$ and

$T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, $T_5(x)$



Example: Again consider,

$f(x) = e^x$ based at $b = 0$

Find the first value of *n* when Taylor's inequality gives an error less than 0.0001 on $[-2,2]$.

Side Note:
For a fixed constant, *a*, the expression $\frac{a^k}{k!}$ goes to zero as k goes to infinity.

So the expression $\frac{1}{(n+1)!}|x-b|^{n+1}$, will always go to zero as *n* gets bigger.

Which means that the error goes to zero, unless something unusual is happening with M, which will see in examples later.

TN 4: Taylor Series

Def'n: The **Taylor Series** for f(x) based at b is defined by

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n\to\infty} T_n(x)$$

If the limit exists at *x*,
then we say it **converges** at x.
(*i.e.* the error goes to zero at x)

Otherwise, we say it **diverges** at x.

The **open interval of convergence** gives the largest open interval over which the series converges.

Note: If

$$\lim_{n\to\infty} \frac{M}{(n+1)!} |x-b|^{n+1} = 0$$

then *x* is in the open interval of convergence.